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SU(3) in an O(3) basis II. Solution of the state labelling problem

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Abstract. The problem of l degeneracies in the reduction of irreducible representations (p, q) of SU(3) with respect to its O(3) subgroup is considered. Orthonormal states corresponding to the same l value are defined as eigenvectors of hermitian operators and, by the use of l shift operators, an algorithm is given for the calculation of the eigenvalues of these hermitian operators. A full analysis is given of the $(p, 0)$ representations, which contain no l degeneracies, and all eigenvalues of the third and fourth order O(3) scalar operators are calculated for the (4, 2), (5, 2) and (6, 2) representations which contain doubly l degenerate states.

1. Introduction

In a previous paper (Hughes 1973a, to be referred to as I) the notation of which will be employed here, the operators $O_l^{\pm 1}$ and $O_l^{\pm 2}$ were constructed from the generators of SU(3). These shift the eigenvalues of l, which labels the irreducible representations of the O(3) subgroup by, respectively, ± 1 and ± 2. The hermiticity properties of these operators were discussed, and expressions for various l commuting products given in terms of the SU(3) invariants I_2, I_3 and the O(3) scalar operators O_l^0 and Q_l^0.

In this paper we shall use these shift operators to obtain an algorithm for the calculation of the eigenvalues of O_l^0 and Q_l^0 thereby, since these latter operators are hermitian, giving an orthogonal solution to the state labelling problem.

The irreducible representations of SU(3) are labelled by the integers (p, q) satisfying $p \geqslant q \geqslant 0$ (Baird and Biedenharn 1963) and related to I_2 and I_3 by equations (1) and (2) of I. The l content of (p, q) is already well known (Elliot 1958a, b, Bargmann and Moshinsky 1960, 1961, De Baenst-van den Broucke *et al* 1970), p being the maximum l value and n, the greatest degree of l degeneracy occurring, being equal to the largest integer less than or equal to $\frac{1}{2}q+1$. The exact l content of (p, q) is shown in figures 1 and 2 for, respectively, even and odd q. The problem of classifying and distinguishing between orthogonal states corresponding to the same degenerate l value has not, however, been completely solved (Racah 1962), although Bargmann and Moshinsky (1961) showed how in principle the eigenvalues of O_l^0 could be calculated using expressions they obtained for its matrix elements between non-orthogonal states. An explicit expression for its eigenvalues was, however, obtained only for the non-degenerate $(p, 0)$ representations, and this agrees with equation (61) given here. It is with respect to this state labelling problem that new results will be obtained here.

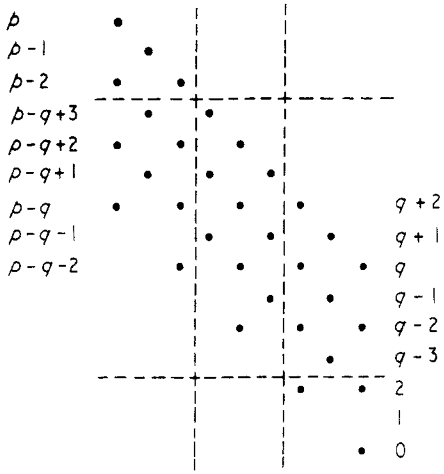

Figure 1. Decomposition of the SU(3) representation (p, q) into O(3) multiplets for even values of q. Each full circle represents an independent multiplet whose l value is given on the left or right of the diagram. The bottom right hand corner depicts multiplets for even p; if p is odd the $l = 0$ multiplet is replaced by an $l = 1$ multiplet, and only one $l = 2$ multiplet occurs.

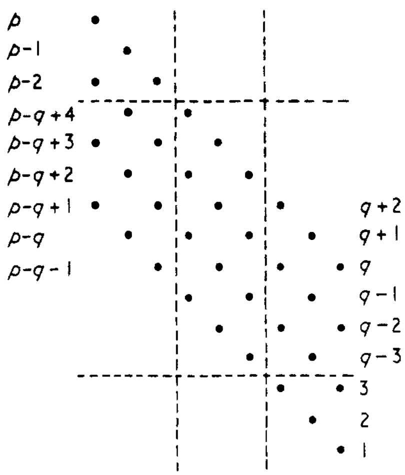

Figure 2. Decomposition of the SU(3) representation (p, q) into O(3) multiplets for odd values of q.

Although the maximum l value of (p, q) is already known, it will be rederived in § 2 since this will provide a simple illustration of the way in which the shift operators work. In so doing the previously uncalculated eigenvalues of O_l^0 and Q_l^0 will be obtained for the maximum l state, together with those of various products of the shift operators which will be needed to proceed to states of lower l values.

In § 3 we shall continue the analysis of (p, q) as far as the $(p-4)$ states. The method by which all states of the representation may be classified will then be clear, although we shall not be able to give the matrix elements of operators for states beyond $l = p-4$. Unfortunately these matrix elements become increasingly laborious as the l value decreases, and it is unlikely that general expressions exist giving them for arbitrary p and q. However it will be clear how one may proceed to a full analysis of any representation corresponding to fixed values of p and q, although for any but very low values of these parameters, such an analysis would be enormously laborious and probably require a computer for its execution. The solution to the state labelling problem given here will therefore be in the nature of an algorithm rather than an exact solution for arbitrary representations; an exact solution probably does not exist.

The matrix elements calculated in § 3 will be sufficient for a complete analysis of the double l degenerate (4, 2), (5, 2) and (6, 2) representations. Since it would be laborious to write down all the matrix elements for these cases, we shall content ourselves by giving only a table of the eigenvalues of O_l^0 and Q_l^0 in the Appendix, together with those for the non-l degenerate adjoint representation (2, 1). A list of all the matrix elements for these representations is given by Hughes (1971).

Since the $(p, 0)$ contain no l degeneracies, a full analysis of these representations is possible. In § 4 we therefore obtain for arbitrary p the eigenvalues of O_l^0 and Q_l^0, and the actions of the shift operators on states of arbitrary l for these representations. In § 5 we discuss various extensions of the present work.

Throughout this paper, attention is restricted to states corresponding to zero m, the eigenvalue of the diagonal O(3) generator l_0. Since, as mentioned in I, the eigenvalues of O_l^0 and Q_l^0 are independent of m, no great loss of generality and a good deal of simplification results from doing this. It also eliminates from our considerations the internal structure of O(3) representations, which is not relevant to the state labelling problem.

2. The maximum l state of (p, q)

Before considering the maximum l state, we first note that since the representations (p, q) and $(p, p-q)$ are mutually contragredient, all formulae relating to $(p, p-q)$ may be obtained from those of (p, q) merely by changing the sign of all odd-ordered operators, such as O_l^0 and I_3, leaving unaltered the signs of even-ordered operators like L^2, Q_l^0 and I_2; clearly the l contents of (p, q) and $(p, p-q)$ are exactly the same. For this reason it suffices to consider representations (p, q) where $q \leqslant [\frac{1}{2}p]$, the largest integer which is less than or equal to $\frac{1}{2}p$; these are the representations for which the eigenvalues of I_3 are greater than or equal to zero.

As long as no confusion is possible as to which representations we are considering, we shall suppress p and q in the basis vectors; since we are considering states corresponding to $m = 0$, we shall simply denote them by $|l, a_l\rangle$ or, if l is non-degenerate, by $|l\rangle$. The formulae for the matrix elements and eigenvalues of the double and triple product operators will be valid only when $m = 0$, but those relating to the O(3) scalars O_l^0 and Q_l^0 will be valid for all m and therefore perfectly general.

The following property of the double product operators will be extremely useful: suppose $\langle l, a_l|O_{l+1}^{-1}O_l^{+1}|l, a_l\rangle = 0$. Then from equation (I, 28), $\langle l+1, b_{l+1}|O_l^{+1}|l, a_l\rangle$ vanishes for all states $|l+1, b_{l+1}\rangle$, so $O_l^{+1}|l, a_l\rangle = 0$. On the other hand, equation (I, 28) also implies that $\langle l, a_l|O_{l+1}^{-1}|l+1, b_{l+1}\rangle$ vanishes for all $|l+1, b_{l+1}\rangle$, so we can never obtain $|l, a_l\rangle$ by the application of O_{l+1}^{-1} to an $(l+1)$ state. Similar considerations hold for the other double product operators.

Now let the maximum value of l occurring in the representations (p, q) be $\bar{l}$ and $|\bar{l}\rangle$ a corresponding state (there is no need to assume that $\bar{l}$ is non-degenerate). Since states

$|\bar{l}+1\rangle$ and $|\bar{l}+2\rangle$ do not exist we must have

$$O_{\bar{l}}^{+1}|\bar{l}\rangle = 0, \qquad O_{\bar{l}}^{+2}|\bar{l}\rangle = 0 \tag{1}$$

which immediately imply that

$$O_{\bar{l}+1}^{-1}O_{\bar{l}}^{+1}|\bar{l}\rangle = 0, \qquad O_{\bar{l}+2}^{-2}O_{\bar{l}}^{+2}|\bar{l}\rangle = 0. \tag{2}$$

From the above-mentioned property of double product operators we see that (2) are sufficient as well as necessary conditions for (1).

Substituting (2) in the expressions (I, 45) and (I, 46) for $Q_{\bar{l}}^0$ and $(O_{\bar{l}}^0)^2$, we obtain

$$Q_{\bar{l}}^0|\bar{l}\rangle = -6(\bar{l}+1)\{12(2\bar{l}+3)I_2-\bar{l}(2\bar{l}^2+14\bar{l}+3)\}|\bar{l}\rangle \tag{3}$$

and

$$O_{\bar{l}}^0|\bar{l}\rangle = \pm 3(\bar{l}+1)(2\bar{l}+3)\{24I_2-2\bar{l}(\bar{l}+4)\}^{1/2}|\bar{l}\rangle \tag{4}$$

where the sign for $O_{\bar{l}}^0$ is as yet undetermined. These imply that $Q_{\bar{l}}^0$ and $O_{\bar{l}}^0$ are diagonal operators with respect to $|\bar{l}\rangle$, and consequently that $[Q_{\bar{l}}^0, O_{\bar{l}}^0]|\bar{l}\rangle = 0$. Using equation (I, 42) we can now calculate $O_{\bar{l}-1}^{+1}O_{\bar{l}+1}^{-2}O_{\bar{l}}^{+1}|\bar{l}\rangle$ in terms of I_2, I_3 and $\bar{l}$: since $O_{\bar{l}}^{+1}|\bar{l}\rangle = 0$, this must vanish so we consequently obtain the following equations to be solved for $\bar{l}$:

$$18\sqrt{6}\, I_3 = \pm\{24I_2-2\bar{l}(\bar{l}+4)\}^{1/2}\{(\bar{l}+1)(\bar{l}+3)-3I_2\} \tag{5}$$

where the $\pm$ correspond to those appearing in equation (4) for $O_{\bar{l}}^0$. Squaring this and using the expressions (I, 1) and (I, 2) for I_2 and I_3 in terms of p and q we arrive at the equation

$$(\bar{l}-p)(\bar{l}+p+4)(\bar{l}-q+1)(\bar{l}+q+3)(\bar{l}-p+q+1)(\bar{l}+p-q+3) = 0. \tag{6}$$

The possible values of $\bar{l}$ are given by the roots of this equation. Clearly the only possibilities consistent with $\bar{l} \geqslant 0$ and $p \geqslant q \geqslant 0$ are $\bar{l} = p, p-q-1$ and $q-1$.

In order to eliminate two of these possibilities we use the criterion that $O_{\bar{l}-1}^{+1}O_{\bar{l}}^{-1}$ be a positive definite operator. Now from equation (I, 47) we have

$$O_{\bar{l}-1}^{+1}O_{\bar{l}}^{-1}|\bar{l}\rangle = -24\bar{l}^2(\bar{l}+1)^2(2\bar{l}+1)\{9I_2-\bar{l}(\bar{l}+3)\}|\bar{l}\rangle,$$

so clearly $9I_2-\bar{l}(\bar{l}+3) \leqslant 0$. When $\bar{l} = p-q-1$, $9I_2-\bar{l}(\bar{l}+3) = (p+1)(q+2) \geqslant 0$, which is not permissible and when $\bar{l} = q-1$, $9I_2-\bar{l}(\bar{l}+3) = (p+1)(p-q+2) \geqslant 0$, also not permissible. However, when $\bar{l} = p$, $9I_2-\bar{l}(\bar{l}+3) = q(q-p) \leqslant 0$, so the maximum value of l occurring in the representation (p, q) can only be $\bar{l} = p$. To prove that (p, q) does in fact have a mximum we should need to make further use of the hermiticity properties of the shift operators, since we have not eliminated the possibility that an $|\bar{l}+1\rangle$ state could still be obtained by the application of $O_{\bar{l}-1}^{+2}$ to $|\bar{l}-1\rangle$. We shall not do this, however, since SU(3) is a compact group and its unitary representations must therefore all be finite dimensional; it is in fact known (Baird and Biedenharn 1963) that (p, q) has dimension $\frac{1}{2}\{(p-q+1)(p+2)(q+1)\}$. $\bar{l}$ must therefore exist, and from the above it can only be p.

We can now determine the correct sign for $O_{\bar{l}}^0$: equation (5) contains the factor $(\bar{l}+1)(\bar{l}+3)-3I_2 = \frac{1}{3}(p+q+3)(2p-q+3)$, which is positive. As mentioned earlier we need consider only values of $q \leqslant [\frac{1}{2}p]$; since therefore I_3 is positive we must clearly take the positive sign in (5) and hence also in (4). Had we considered the contragredients of such representations, for which $q \geqslant [\frac{1}{2}p]$, I_3 would be negative and we should have to take the negative square root for O_p^0. This of course fits in with the fact that on passing from (p, q) to its contragredient $(p, p-q)$ odd-ordered operators like O_l^0 change sign.

In this respect it is worth noting that equation (6) is symmetric under the interchange of q and $p-q$, a feature which will be shared by all other formulae obtained in this paper.

To summarize, apart from obtaining the already well-known fact that p is the maximum l value we have also obtained the actions of Q_p^0 and O_p^0 on the maximum l state $|p\rangle$. These, together with the actions of other product operators which will be needed in the subsequent analysis, are

$$Q_p^0|p\rangle = -2(p+1)\{2p^3-2(4q+3)p^2+(8q^2-12q+27)p+12q^2\}|p\rangle \tag{7}$$

$$O_p^0|p\rangle = \surd 6(p+1)(2p+3)(p-2q)|p\rangle \tag{8}$$

$$O_{p-1}^{+1}O_p^{-1}|p\rangle = 24p^2(p+1)^2(2p+1)(p-q)q|p\rangle \tag{9}$$

$$O_{p-2}^{+2}O_p^{-2}|p\rangle = 48p^3(p-1)^2(2p+1)(p^3-2p^2+(1-2q)p+2q^2)|p\rangle \tag{10}$$

$$O_{p-1}^{+1}O_{p-2}^{+1}O_p^{-2}|p\rangle = 96\surd 6(2p+1)p^3(p+1)^2(p-1)^2(p-2q)(p-q)q|p\rangle. \tag{11}$$

Equations (9) and (10) were obtained using (I, 47) and (I, 48), whereas (11) was obtained by substituting O_p^0 and Q_p^0 into (I, 40).

3. Further analysis of (p, q)

The next step in our analysis is to consider the $l = p-1$ state defined by $|p-1\rangle \propto O_p^{-1}|p\rangle$. From (9) we see that this exists provided $q \neq 0$; since we shall treat the case $q = 0$ separately in § 4 we exclude it here and assume $|p-1\rangle$ to exist. It is already well known, and can be seen by inspecting figures 1 and 2, that $l = p-1$ is non-degenerate; we shall not therefore give a rigorous proof here, and content ourselves with noting that since $|p+1\rangle$ does not exist there is no possibility of a different $(p-1)$ state arising from $O_{p+1}^{-2}|p+1\rangle$. Also, as mentioned in § 2, the assumption that $l = p$ is non-degenerate is also not necessary. To prove it rigorously we should have to show at every step of the analysis that in obtaining an l state from a state of higher l by the application of a lowering shift operator of one kind, it is not possible to start a different chain of states of increasing l values by the application of the raising shift operator of the other kind. However, since the non-degeneracy of $l = p$, and in fact the degeneracy of every l value, is well known, we shall save space by omitting such a proof.

First, since $|p+1\rangle$ does not exist, we have

$$O_{p+1}^{-2}O_{p-1}^{+2}|p-1\rangle = 0. \tag{12}$$

Next, using the non-degeneracy of $l = p$ and $l = p-1$ in (I, 30), we have

$$\langle p|O_{p-1}^{+1}O_p^{-1}|p\rangle = \langle p-1|O_p^{-1}O_{p-1}^{+1}|p-1\rangle,$$

so from equation (9) we obtain

$$O_p^{-1}O_{p-1}^{+1}|p-1\rangle = 24p^2(p+1)^2(2p+1)(p-q)q|p-1\rangle, \tag{13}$$

and then from (I, 45) and (I, 46) we get

$$\begin{aligned} Q_{p-1}^0|p-1\rangle = {} & -2\{2p^4-4(2q-1)p^3+(8q^2-28q+69)p^2+(28q^2-12q-27)p \\ & +12q^2\}|p-1\rangle, \end{aligned} \tag{14}$$

$$(O_{p-1}^0)^2|p-1\rangle = 6(p+3)^2(2p+1)^2(p-2q)^2|p-1\rangle.$$

Substituting $O_p^{-1}O_{p+1}^{-1}O_{p-1}^{+2}|p-1\rangle = 0$ in (I, 39) and using the above expressions for

$Q^0_{p-1}|p-1\rangle$ and $(O^0_{p-1})^2|p-1\rangle$ yields an equation giving $O^0_{p-1}|p-1\rangle$ in terms of $I_3|p-1\rangle$ whose solution is

$$O^0_{p-1}|p-1\rangle = \surd 6(p+3)(2p+1)(p-2q)|p-1\rangle. \tag{15}$$

The following formulae will be needed in the subsequent analysis:

$$O^{+1}_{p-2}O^{-1}_{p-1}|p-1\rangle = 48p(p+1)(p-1)^2\{2(q-1)p^2-(2q^2-2q-1)p-2q^2\}|p-1\rangle \tag{16}$$

$$O^{+2}_{p-3}O^{-2}_{p-1}|p-1\rangle = 48(2p-1)p(p-1)^2(p-2)^2\{p^3-6p^2-(6q-17)p+6(q^2-2)\}|p-1\rangle \tag{17}$$

$$\begin{aligned} O^{-1}_p O^{+2}_{p-2}O^{-1}_{p-1}|p-1\rangle &= O^{+1}_{p-2}O^{-2}_p O^{+1}_{p-1}|p-1\rangle \\ &= 96\surd 6(2p+1)(p+1)^2p^3(p-1)^2(p-2q)q(p-q)|p-1\rangle \end{aligned} \tag{18}$$

$$\begin{aligned} O^{+1}_{p-2}O^{+1}_{p-3}O^{-2}_{p-1}|p-1\rangle &= 96\surd 6(2p-1)(p+1)p(p-1)^2(p-2)^2(p-2q) \\ &\quad \times\{(3q-2)p^2-(3q^2-3q+4)p-3(q^2-2)\}|p-1\rangle. \end{aligned} \tag{19}$$

Equations (16), (17) and (19) were obtained using (I, 47), (I, 48) and (I, 40); (18) was obtained using the hermiticity relation (I, 34), and (I, 31) which implies the equality of

$$\langle p-1|O^{+1}_{p-2}O^{-2}_p O^{+1}_{p-1}|p-1\rangle \qquad \text{and} \qquad \langle p|O^{+1}_{p-1}O^{+1}_{p-2}O^{-2}_p|p\rangle.$$

We can now also obtain the matrix elements of O^{-1}_p and O^{+1}_{p-1}. From (I, 29) we have

$$\langle p|O^{+1}_{p-1}O^{-1}_p|p\rangle = \alpha_{1,p-1}|\langle p-1|O^{-1}_p|p\rangle|^2 = \frac{1}{\alpha_{1,p-1}}|\langle p|O^{+1}_{p-1}|p-1\rangle|^2.$$

We shall use the phase convention that $\langle p-1|O^{-1}_p|p\rangle$ be real and non-negative; this implies that $\langle p|(O^{-1}_p)^\dagger|p-1\rangle$ and therefore also, by (I, 23), $\langle p|O^{+1}_{p-1}|p-1\rangle$ are real and non-negative. Using (9) we obtain

$$O^{-1}_p|p\rangle = 2\surd 6(2p+1)(p+1)p\left(\frac{q(p-q)}{2p-1}\right)^{1/2}|p-1\rangle \tag{20}$$

$$O^{+1}_{p-1}|p-1\rangle = 2\surd 6(p+1)p\{(2p-1)q(p-q)\}^{1/2}|p\rangle. \tag{21}$$

The next stage is to consider states corresponding to $l=(p-2)$ and here we have to deal with the problem of degeneracy since they can be arrived at in two independent ways, namely from $O^{-2}_p|p\rangle$ and $O^{-1}_{p-1}|p-1\rangle$. Our aim is to define two mutually orthogonal states $|p-2, 1\rangle$ and $|p-2, 2\rangle$, which could be done by defining them to be eigenvectors of O^0_{p-2} or Q^0_{p-2}. However, it would be difficult to do this immediately and instead we define them using the shift operators. $|p-2, 1\rangle$ could be defined as the normalized state obtained from either $O^{-2}_p|p\rangle$ or $O^{-1}_{p-1}|p-1\rangle$. Neither of these choices is particularly more convenient than the other and we choose to define $|p-2, 1\rangle \propto O^{-2}_p|p\rangle$. Now since

$$O^{+2}_{p-2}O^{-2}_p|p\rangle \propto |p\rangle, \qquad O^{+2}_{p-2}|p-2, 1\rangle \propto |p\rangle$$

and so $O^{-2}_p O^{+2}_{p-2}|p-2, 1\rangle \propto |p-2, 1\rangle$; hence $|p-2, 1\rangle$ is an eigenvector of $O^{-2}_p O^{+2}_{p-2}$. Using $\langle p-2, 1|O^{-2}_p O^{+2}_{p-2}|p-2, 1\rangle = \langle p|O^{+2}_{p-2}O^{-2}_p|p\rangle$ and (10), we obtain

$$O^{-2}_p O^{+2}_{p-2}|p-2, 1\rangle = 48(2p+1)p^3(p-1)^2\{p^3-2p^2+(1-2q)p+2q^2\}|p-2, 1\rangle. \tag{22}$$

We can also easily deduce that

$$O_p^{-2}|p\rangle = 4\sqrt{3}p(p-1)(2p+1)\left(\frac{p\{p^3-2p^2+(1-2q)p+2q^2\}}{2p-3}\right)^{1/2}|p-2,1\rangle \tag{23}$$

$$O_{p-2}^{+2}|p-2,1\rangle = 4\sqrt{3}p(p-1)\{p(2p-3)(p^3-2p^2+(1-2q)p+2q^2)\}^{1/2}|p\rangle. \tag{24}$$

Now define $|p-2,2\rangle$ to be a normalized state orthogonal to $|p-2,1\rangle$ (its phase will be chosen later). By hermiticity we have

$$\langle p|O_{p-2}^{+2}|p-2,2\rangle = \alpha_{2,p-2}\langle p-2,2|O_p^{-2}|p\rangle^* = 0,$$

so $O_{p-2}^{+2}|p-2,2\rangle$ is orthogonal to $|p\rangle$ and therefore, since $l = p$ is non-degenerate, $O_{p-2}^{+2}|p-2,2\rangle = 0$. This then implies that

$$O_p^{-2}O_{p-2}^{+2}|p-2,2\rangle = 0. \tag{25}$$

From (22) and (25) we see that $|p-2,1\rangle$ and $|p-2,2\rangle$ correspond to distinct eigenvalues of the hermitian operator $O_p^{-2}O_{p-2}^{+2}$; this then guarantees their orthogonality.

We next obtain the actions $O_{p-1}^{-1}O_{p-2}^{+1}$ on $|p-2,1\rangle$ and $|p-2,2\rangle$. Suppose

$$O_{p-1}^{-1}|p-1\rangle = a|p-2,1\rangle + b|p-2,1\rangle. \tag{26}$$

Then by hermiticity we have

$$O_{p-2}^{+1}|p-2,1\rangle = \alpha_{1,p-2}a^*|p-1\rangle, \qquad O_{p-2}^{+1}|p-2,2\rangle = \alpha_{1,p-2}b^*|p-1\rangle. \tag{27}$$

For convenience denote

$$O_p^{-1}|p\rangle = \gamma|p-1\rangle, \qquad O_{p-2}^{+2}|p-2,1\rangle = \delta|p\rangle,$$
$$O_p^{-1}O_{p-2}^{+2}O_{p-1}^{-1}|p-1\rangle = A|p-1\rangle, \qquad O_{p-2}^{+1}O_{p-1}^{-1}|p-1\rangle = B|p-1\rangle,$$

where γ, δ, A and B are already written down in this paper. Now

$$O_p^{-1}O_{p-2}^{+2}O_{p-1}^{-1}|p-1\rangle = aO_p^{-1}O_{p-2}^{+2}|p-2,1\rangle + bO_p^{-1}O_{p-2}^{+2}|p-2,2\rangle = a\delta\gamma|p-1\rangle,$$

so $a = A/\gamma\delta$. Also

$$O_{p-2}^{+1}O_{p-1}^{-1}|p-1\rangle = aO_{p-2}^{+1}|p-2,1\rangle + bO_{p-2}^{+1}|p-2,2\rangle = \alpha_{1,p-2}(|a|^2+|b|^2)|p-1\rangle,$$

so

$$|b|^2 = \frac{B}{\alpha_{1,p-2}} - |a|^2.$$

We see that whereas a has already been chosen real and non-negative, the phase of b is still undetermined, so we choose it to be also real and non-negative. Using the expressions for γ, δ, A, B and $\alpha_{1,p-2}$ we readily obtain a and b which on substitution in (26) and (27) give

$$O_{p-1}^{-1}|p-1\rangle = 4\sqrt{3}(p-1)(p+1)(p-2q)\left(\frac{(2p-1)pq(p-q)}{(2p-3)\{p^3-2p^2+(1-2q)p+2q^2\}}\right)^{1/2}|p-2,1\rangle$$
$$+4\sqrt{3}(2p-1)p(p-1)\left(\frac{(p+1)p(p-1)(q-1)(p-q-1)}{(2p-3)\{p^3-2p^2+(1-2q)p+2q^2\}}\right)^{1/2}|p-2,2\rangle \tag{28}$$

$$O^{+1}_{p-2}|p-2, 1\rangle$$
$$= 4\sqrt{3(p-1)(p+1)(p-2q)}\left(\frac{(2p-3)pq(p-q)}{(2p-1)\{p^3-2p^2+(1-2q)p+2q^2\}}\right)^{1/2}|p-1\rangle \tag{29}$$

$$O^{+1}_{p-2}|p-2, 2\rangle = 4\sqrt{3p(p-1)}\left(\frac{(2p-3)(p+1)p(p-1)(q-1)(p-q-1)}{p^3-2p^2+(1-2q)p+2q^2}\right)^{1/2}|p-1\rangle. \tag{30}$$

Using these equations we may obtain the following equations:

$$O^{-1}_{p-1}O^{+1}_{p-2}|p-2, 1\rangle$$
$$= \frac{48(p-1)^2(p+1)^2(p-2q)^2pq(p-q)}{p^3-2p^2+(1-2q)p+2q^2}|p-2, 1\rangle + \frac{48p^2(p-1)^2(p+1)(p-2q)}{p^3-2p^2+(1-2q)p+2q^2}$$
$$\times\{(2p-1)(p+1)(p-1)q(q-1)(p-q)(p-q-1)\}^{1/2}|p-2\ 2\rangle \tag{31}$$

$$O^{-1}_{p-1}O^{+1}_{p-2}|p-2, 2\rangle$$
$$= \frac{48p^2(p-1)^2(p+1)(p-2q)}{p^3-2p^2+(1-2q)p+2q^2}$$
$$\times\{(2p-1)(p+1)(p-1)q(q-1)(p-q)(p-q-1)\}^{1/2}|p-2, 1\rangle$$
$$+\frac{48p^3(p-1)^3(2p-1)(p+1)(q-1)(p-q-1)}{p^3-2p^2+(1-2q)p+2q^2}|p-2, 2\rangle. \tag{32}$$

We are now in a position to calculate the actions of $O^{+1}_{p-3}O^{-1}_{p-2}$, $O^{+2}_{p-4}O^{-2}_{p-2}$, $(O^0_{p-2})^2$ and Q^0_{p-2} on $|p-2, 1\rangle$ and $|p-2, 2\rangle$. The expressions for these actions are rather laborious to write out for the general case and so we shall omit them here; for the $(p, 2)$ representations they are given by Hughes (1971). In defining the $l = p-3$ and $l = p-4$ states we shall be content to give the general method of writing down the matrix elements of the various operators and omit their explicit expressions. The matrix elements of O^0_{p-2} can be calculated from those of $(O^0_{p-2})^2$ and $O^{-1}_{p-1}O^{-1}_pO^{+2}_{p-2}$, the latter being obtained from the already given expressions for $O^{+2}_{p-2}|p-2, 1\rangle$, $O^{+2}_{p-2}|p-2, 2\rangle$, $O^{-1}_p|p\rangle$ and $O^{-1}_{p-1}|p-1\rangle$. At the end of this section we shall discuss the general problem of obtaining matrix elements and eigenvalues of O^0_l in cases of l degeneracy, and in the Appendix write the eigenvalues down for the (4, 2), (5, 2) and (6, 2) representations.

Passing first to the $l = (p-3)$ states, we define $|p-3, 1\rangle$ to be the state obtained by application of O^{-2}_{p-1} to $|p-1\rangle$ and $|p-3, 2\rangle$ to be the state annihilated by O^{+2}_{p-3}. We then obtain, in a manner entirely analogous to that used for the $(p-2)$ states, the following actions:

$$O^{-2}_{p-1}O^{+2}_{p-3}|p-3, 1\rangle$$
$$= 48(2p-1)p(p-1)^2(p-2)^2\{p^3-6p^2-(6q-17)p+6(q^2-2)\}|p-3, 1\rangle \tag{33}$$

$$O^{-2}_{p-1}|p-1\rangle = 4\sqrt{3(2p-1)(p-1)(p-2)}\left(\frac{p\{p^3-6p^2-(6q-17)p+6(q^2-2)\}}{2p-5}\right)^{1/2}|p-3, 1\rangle \tag{34}$$

$$O^{+2}_{p-3}|p-3, 1\rangle = 4\sqrt{3(p-1)(p-2)}[p(2p-5)\{p^3-6p^2-(6q-17)p+6(q^2-2)\}]^{1/2}|p-1\rangle \tag{35}$$

$$O^{-2}_{p-1}O^{+2}_{p-3}|p-3, 2\rangle = 0. \tag{36}$$

Equations (33) and (36) show that $|p-3,1\rangle$ and $|p-3,2\rangle$ are eigenvectors of the hermitian operator $O_{p-1}^{-2}O_{p-3}^{+2}$ corresponding to distinct eigenvalues, and so must be orthogonal. $|p-3,1\rangle$ is normalized and its phase has been determined by the requirement that O_{p-1}^{-2} have a real non-negative matrix element between $|p-1\rangle$ and $|p-3,1\rangle$. The phase of $|p-3,2\rangle$ is as yet undetermined.

Now we show how to obtain the actions of $O_{p-2}^{-1}O_{p-3}^{+1}$ on $|p-3,1\rangle$ and $|p-3,2\rangle$. This is rather more complicated than for the $(p-2)$ states since we now have two $(p-2)$ states as well as the two $(p-3)$ states. There are two methods of doing this, the first method being to use the matrix elements of $O_{p-3}^{+1}O_{p-2}^{-1}$ and $O_{p-2}^{+1}O_{p-3}^{+1}O_{p-1}^{-2}$. This method, which is given in detail by Hughes (1971), suffers from the disadvantage that although the above matrix elements are relatively simple to calculate, the resulting matrix elements of O_{p-2}^{-1} involve sign ambiguities which in general cannot be solved, although for the $(p, 2)$ representations these ambiguities do not arise. This disadvantage is not present in the second method, which we now discuss.

Suppose

$$\begin{aligned} O_{p-2}^{-1}|p-2,1\rangle &= a_1|p-3,1\rangle + b_1|p-3,2\rangle \\ O_{p-2}^{-1}|p-2,2\rangle &= a_2|p-3,1\rangle + b_2|p-3,2\rangle, \end{aligned} \tag{37}$$

so that by hermiticity

$$\begin{aligned} O_{p-3}^{+1}|p-3,1\rangle &= \alpha_{1,p-3}(a_1^*|p-2,1\rangle + a_2^*|p-2,2\rangle) \\ O_{p-3}^{+1}|p-3,2\rangle &= \alpha_{1,p-3}(b_1^*|p-2,1\rangle + b_2^*|p-2,2\rangle). \end{aligned} \tag{38}$$

We denote

$$O_{p-2}^{+1}|p-2,1\rangle = \eta_1|p-1\rangle, \qquad O_{p-2}^{+1}|p-2,2\rangle = \eta_2|p-1\rangle$$
$$O_{p-1}^{-2}|p-1\rangle = \epsilon|p-3,1\rangle,$$

all of which have been given in this section, and

$$\langle p-2,i|O_{p-1}^{-1}O_{p-3}^{+2}O_{p-2}^{-1}|p-2,j\rangle = A_{ji}(i,j=1,2).$$

The A_{ji} can be calculated from formulae already derived in this section and (I, 41), since from the known expressions for the actions of $O_p^{-2}O_{p-2}^{+2}$ and $O_{p-1}^{-1}O_{p-2}^{+1}$ on $|p-2,1\rangle$ and $|p-2,2\rangle$ we may calculate those of Q_{p-2}^0, $(O_{p-2}^0)^2$ and, with some difficulty, O_{p-2}^0.

Now since $\langle p-2,i|O_p^{-2}O_{p-2}^{+2}|p-2,j\rangle$ and $\langle p-2,i|O_{p-1}^{-1}O_{p-2}^{+1}|p-2,j\rangle$ are real, so are $\langle p-2,i|Q_{p-2}^0|p-2,j\rangle$ and $\langle p-2,i|(O_{p-2}^0)^2|p-2,j\rangle$. One may easily deduce that $\langle p-2,i|O_{p-2}^0|p-2,j\rangle$ are also real, so finally the A_{ji} are all real. Now

$$\begin{aligned} A_{11}|p-2,1\rangle + A_{12}|p-2,2\rangle &= O_{p-1}^{-1}O_{p-3}^{+2}O_{p-2}^{-1}|p-2,1\rangle \\ &= a_1 O_{p-1}^{-1}O_{p-3}^{+2}|p-3,1\rangle + b_1 O_{p-1}^{-1}O_{p-3}^{+2}|p-3,2\rangle \\ &= a_1 O_{p-1}^{-1}O_{p-3}^{+2}|p-3,1\rangle \\ &= a_1\alpha_{2,p-3}\epsilon O_{p-1}^{-1}|p-1\rangle \\ &= \frac{a_1\alpha_{2,p-3}\epsilon}{\alpha_{1,p-2}}(\eta_1|p-2,1\rangle + \eta_2|p-2,2\rangle). \end{aligned}$$

But $\alpha_{2,p-3}/\alpha_{1,p-2} = \alpha_{1,p-3}$, so using the orthogonality of $|p-2, 1\rangle$ and $|p-2, 2\rangle$, we obtain

$$a_1 = A_{11}/\epsilon\eta_1\alpha_{1,p-3} = A_{12}/\epsilon\eta_2\alpha_{1,p-3}.$$

By considering $O^{-1}_{p-1}O^{+2}_{p-3}O^{-1}_{p-2}|p-2, 2\rangle$ we obtain also

$$a_2 = A_{22}/\epsilon\eta_2\alpha_{1,p-3} = A_{21}/\epsilon\eta_1\alpha_{1,p-3}.$$

We see from this that in fact only two of the A_{ji} need to be known. Note that the a_i are real. We still have to calculate b_1 and b_2; denote

$$\langle p-2, i|O^{+1}_{p-3}O^{-1}_{p-2}|p-2, j\rangle = B_{ji} \qquad (i = 1, 2).$$

Then

$$\begin{aligned}
&B_{11}|p-2, 1\rangle + B_{12}|p-2, 2\rangle \\
&\qquad = O^{+1}_{p-3}O^{-1}_{p-2}|p-2, 1\rangle \\
&\qquad = a_1 O^{-1}_{p-3}|p-3, 1\rangle + b_1 O^{+1}_{p-3}|p-3, 2\rangle \\
&\qquad = \alpha_{1,p-3}\{a_1(a_1^*|p-2, 1\rangle + a_2^*|p-2, 2\rangle) + b_1(b_1^*|p-2, 1\rangle + b_2^*|p-2, 2\rangle)\},
\end{aligned}$$

so by equating coefficients of $|p-2, 1\rangle$ and $|p-2, 2\rangle$ we obtain $|a_1|^2 + |b_1|^2 = B_{11}/\alpha_{1,p-3}$ and $a_1a_2^* + b_1b_2^* = B_{12}/\alpha_{1,p-3}$. By considering $O^{+1}_{p-3}O^{-1}_{p-2}|p-2, 2\rangle$ we further obtain $|a_2|^2 + |b_2|^2 = B_{22}/\alpha_{1,p-3}$. b_1 and b_2 are determined up to a phase by the first and last of these equations. Since we have not yet chosen the phase of $|p-3, 2\rangle$, we may now choose it so that b_1 is non-negative and real. b_2, which must therefore also be real, is now completely determined by the second of the above equations.

Having obtained a_1, etc, we can calculate in a straightforward manner the actions of $O^{-1}_{p-2}O^{+1}_{p-3}$ on $|p-3, 1\rangle$ and $|p-3, 2\rangle$. Knowing $O^{-2}_{p-1}O^{+2}_{p-3}$ on these states we can then, using (I, 45) to (I, 48), calculate the actions of all other operators including Q^0_{p-3} and, eventually, O^0_{p-3} on $|p-3, 1\rangle$ and $|p-3, 2\rangle$. For the $(p, 2)$ representations (Hughes 1971) considerable simplification occurs in the above calculations since b_1 and b_2 both vanish so that $|p-3, 2\rangle$ does not exist and so $l = (p-3)$ is non-degenerate.

So far we have treated only the simplest case of twofold l degeneracy and shown that our procedure is completey adequate for dealing with such cases. In order to make reasonable the assertion that it works, in principle, for all order degeneracy it is worth giving an outline of the procedure for the next simplest case of threefold l degeneracy. The $l = (p-4)$ states provide us with such an example, and we therefore give a brief discussion of them.

In defining two of the $(p-4)$ states we use the shift operators O^{-2}_{p-2} acting on the $|p-2, i\rangle$. The procedure here is not as straightforward as that used in defining the $(p-2)$ and $(p-3)$ states since if we define

$$|p-4, a\rangle \propto O^{-2}_{p-2}|p-2, 1\rangle \qquad \text{and} \qquad |p-4, b\rangle \propto O^{-2}_{p-2}|p-2, 2\rangle$$

we have no guarantee that they will be orthogonal because they will not in general be eigenvectors of the operator $O^{-2}_{p-2}O^{+2}_{p-4}$. To see this note that from (I, 42) $O^{+2}_{p-4}O^{-2}_{p-2}$ depends on one (and only one) non-diagonal operator, namely $O^{-1}_{p-1}O^{+1}_{p-2}$, so $\langle p-2, 2|O^{+2}_{p-4}O^{-2}_{p-2}|p-2, 1\rangle$ must be non-zero. However,

$$\langle p-2, 2|O^{+2}_{p-4}O^{-2}_{p-2}|p-2, 1\rangle \propto \langle p-2, 2|O^{+2}_{p-4}|p-4, a\rangle.$$

so $\langle p-2, 2|O^{+2}_{p-4}|p-4, a\rangle \neq 0$. Now

$$\begin{aligned}\langle p-4, b|O^{-2}_{p-2}O^{+2}_{p-4}|p-4, a\rangle &= \langle p-4, b|O^{-2}_{p-2}|p-2, 1\rangle\langle p-2, 1|O^{+2}_{p-4}|p-4, a\rangle \\ &\quad + \langle p-4, b|O^{-2}_{p-2}|p-2, 2\rangle\langle p-2, 2|O^{+2}_{p-4}|p-4, a\rangle\end{aligned}$$

and whereas the first of these terms vanishes, the second does not so $O^{-2}_{p-2}O^{+2}_{p-4}$ has a non-vanishing off-diagonal matrix element.

Instead we define $|p-4, 1\rangle$ and $|p-4, 2\rangle$ to be eigenvectors of $O^{-2}_{p-2}O^{+2}_{p-4}$, that is,

$$O^{-2}_{p-2}O^{+2}_{p-4}|p-4, i\rangle = D_{ii}|p-4, i\rangle, \qquad i = 1, 2. \tag{39}$$

Let

$$\begin{aligned}O^{-2}_{p-2}|p-2, 1\rangle &= c_1|p-4, 1\rangle + d_1|p-4, 2\rangle \\ O^{-2}_{p-2}|p-2, 2\rangle &= c_2|p-4, 1\rangle + d_2|p-4, 2\rangle\end{aligned} \tag{40}$$

so that by hermiticity

$$\begin{aligned}O^{+2}_{p-4}|p-4, 1\rangle &= \alpha_{1,p-4}(c_1^*|p-2, 1\rangle + c_2^*|p-2, 2\rangle) \\ O^{+2}_{p-4}|p-4, 2\rangle &= \alpha_{1,p-4}(d_1^*|p-2, 1\rangle + d_2^*|p-2, 2\rangle).\end{aligned} \tag{41}$$

Denote for convenience

$$\begin{aligned}O^{+2}_{p-4}O^{-2}_{p-2}|p-2, 1\rangle &= C_{11}|p-2, 1\rangle + C_{12}|p-2, 2\rangle \\ O^{+2}_{p-4}O^{-2}_{p-2}|p-2, 2\rangle &= C_{12}|p-2, 1\rangle + C_{22}|p-2, 2\rangle,\end{aligned}$$

where the C_{ij} may be assumed known (they can easily be calculated from the expressions for $O^{-2}_{p}O^{+2}_{p-2}|p-2, i\rangle$ and $O^{-1}_{p-1}O^{+1}_{p-2}|p-2, i\rangle$ given earlier on).

By considering the actions of $O^{+2}_{p-4}O^{-2}_{p-2}$ on $|p-2, i\rangle$ one easily obtains

$$|c_1|^2 + |d_1|^2 = \frac{C_{11}}{\alpha_{1,p-4}}, \qquad |c_2|^2 + |d_2|^2 = \frac{C_{22}}{\alpha_{1,p-4}}, \qquad c_1^*c_2 + d_1^*d_2 = \frac{C_{12}}{\alpha_{1,p-4}}, \tag{42}$$

and from the actions of $O^{-2}_{p-2}O^{+2}_{p-4}$ on $|p-4, i\rangle$ follow

$$c_1^*d_1 + c_2^*d_2 = 0, \qquad |c_1|^2 + |c_2|^2 = \frac{D_{11}}{\alpha_{1,p-4}}, \qquad |d_1|^2 + |d_2|^2 = \frac{D_{22}}{\alpha_{1,p-4}}. \tag{43}$$

Equations (42) and the first of (43), which arises because $|p-4, i\rangle$ were chosen to be eigenvectors of $O^{-2}_{p-2}O^{+2}_{p-4}$, can be used to solve for c_1 etc, and the last two of equations (43) may then be used to obtain the eigenvalues of $O^{-2}_{p-2}O^{+2}_{p-4}$.

Elimination of d_1, c_2 and d_2 gives rise to a quadratic equation in $|c_1|^2$, this also being satisfied by $|d_1|^2$. The solutions are

$$|c_1|^2 = \frac{1}{2\alpha_{1,p-4}}\left\{C_{11} + \left(\frac{C_{11}^2(C_{11}-C_{22})^2 + 4C_{12}^2(C_{11}^2 - C_{11}C_{22} + C_{12}^2)}{(C_{11}-C_{22})^2 + 4C_{12}^2}\right)^{1/2}\right\} \tag{44}$$

$$|d_1|^2 = \frac{1}{2\alpha_{1,p-4}}\left\{C_{11} - \left(\frac{C_{11}^2(C_{11}-C_{22})^2 + 4C_{12}^2(C_{11}^2 - C_{11}C_{22} + C_{12}^2)}{(C_{11}-C_{22})^2 + 4C_{12}^2}\right)^{1/2}\right\} \tag{45}$$

($|c_1|^2$ and $|d_1|^2$ could have been interchanged—this would be equivalent to interchanging the roles of $|p-4, 1\rangle$ and $|p-4, 2\rangle$) and

$$|c_2|^2 = \frac{1}{2\alpha_{1,p-4}}\left\{C_{22} - \left(\frac{C_{22}^2(C_{11}-C_{22})^2 + 4C_{12}^2(C_{22}^2 - C_{11}C_{22} + C_{12}^2)}{(C_{11}-C_{22})^2 + 4C_{12}^2}\right)^{1/2}\right\} \tag{46}$$

$$|d_2|^2 = \frac{1}{2\alpha_{1,p-4}}\left\{C_{22} + \left(\frac{C_{22}^2(C_{11}-C_{22})^2 + 4C_{12}^2(C_{22}^2 - C_{11}C_{22} + C_{12}^2)}{(C_{11}-C_{22})^2 + 4C_{12}^2}\right)^{1/2}\right\}. \tag{47}$$

$|p-4, 1\rangle$ and $|p-4, 2\rangle$ can be chosen so that c_1 and d_1 are real and non-negative, and the phases of c_2 and d_2 then determined by requiring compatibility with the equations $c_1c_2 + d_1d_2 = C_{12}/\alpha_{1,p-4}$ and $c_1d_1 + c_2^*d_2 = 0$.

Having determined all these constants, the eigenvalues D_{11} and D_{22} of $O_{p-2}^{-2}O_{p-4}^{+2}$ can then be determined. In general these will be different so that $|p-4, 1\rangle$ and $|p-4, 2\rangle$ will be orthogonal. The definition of the third state is easy, $|p-4, 3\rangle$ being chosen so that O_{p-4}^{+2} annihilates it, that is,

$$O_{p-2}^{-2}O_{p-4}^{+2}|p-4, 3\rangle = 0.$$

This guarantees that $|p-4, 3\rangle$ is orthogonal to the other two states.

The actions of O_{p-3}^{-1} on the $(p-3)$ states and O_{p-4}^{+1} on the $(p-4)$ states can be obtained from the matrix elements of $O_{p-4}^{+1}O_{p-3}^{-1}$ and, for instance, $O_{p-2}^{-1}O_{p-4}^{+2}O_{p-3}^{-1}$, which by this stage will have already been calculated. The actions of all the other operators on the $(p-4)$ states can then also be calculated, enabling one to obtain both the eigenvalues and eigenvectors of Q_{p-4}^0 and O_{p-4}^0 and also to proceed to the consideration of the $(p-5)$ states.

The method of defining states corresponding to arbitrary l values should by now be clear; these definitions, together with the way in which the $O_l^{\pm 2}$ operators interconnect the various states, are illustrated in figure 3. Eventually, as l passes below the value $[\frac{1}{2}p]$,

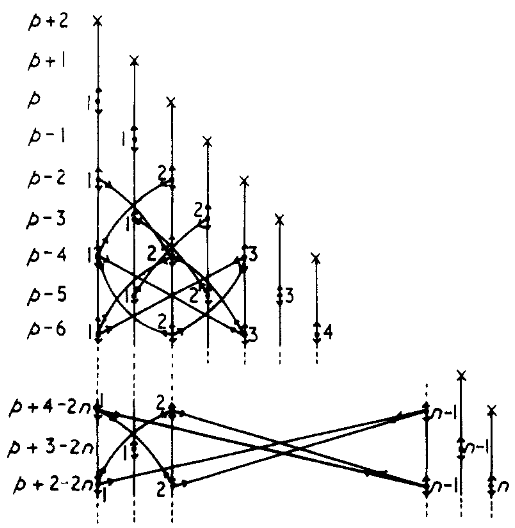

Figure 3. The l values occurring in (p, q). Each full circle represents a state and all states are orthonormal. The operators $O_l^{\pm 2}$ have non-vanishing matrix elements between any two states connected by a single line with arrows. A line with a single arrow ending in a cross indicates that the state from which it originates is annihilated by O_l^{+2}. All states corresponding to consecutive values may also be connected by the $O_l^{\pm 1}$.

its degeneracy will start to decrease again, and this will be effected by the annihilation of states by O_l^{-2}, in a manner parallel to the annihilation, for $l > [\frac{1}{2}p]$, of states by O_l^{+2} This is illustrated by Hughes (1971) for the representations (4, 2), (5, 2), (6, 2) in which the matrix elements of all operators are listed for all states. In this paper we merely illustrate the actions of the shift operators on the states of these representations in, respectively, figures 4, 5 and 6, and give in the Appendix tables of eigenvalues of Q_l^0 and O_l^0 for the three representations. Note that in (4, 2) the $|p-3\rangle$ state does not occur, and neither does $|p-4, 2\rangle$, $|0\rangle$ corresponding to $|p-4, 1\rangle$. On the other hand, in (5, 2)

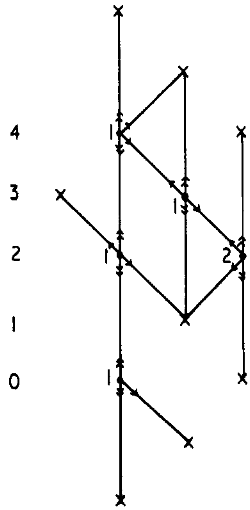

Figure 4. States of (4, 2) plotted against l values. Each state is represented by a full circle and all states are orthonormal; states connected by a line with single or double arrows are connected by, respectively, $O_l^{\pm 1}$ or $O_l^{\pm 2}$. A line connecting a state to a cross indicates that it is annihilated by the corresponding shift operator.

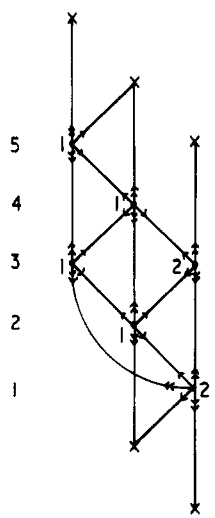

Figure 5. States of (5, 2) plotted against l values. The notation is the same as in figure 4. (5, 3) is represented by exactly the same diagram.

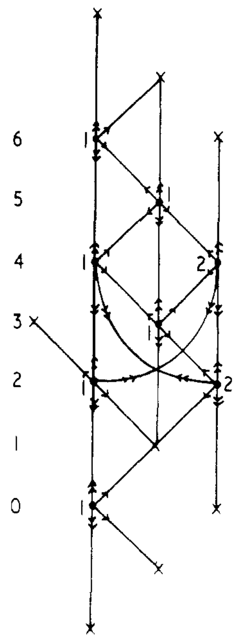

Figure 6. States of (6, 2) plotted against l values. The notation is the same as in figure 4. (6, 4) is represented by exactly the same diagram.

$|p-4, 1\rangle$ does not occur and $|1\rangle$ corresponds to $|p-4, 2\rangle$. All states considered in this section occur in (6, 2).

We end this section with an outline of the method of obtaining the eigenvalues of O_l^0 once those of $(O_l^0)^2$ are known; we shall consider only the case of double l degeneracy, the procedure when the l degeneracy is greater being an obvious extension of this. Let

$$(O_l^0)^2|l, i\rangle = \sum_{j=1}^{2} G_{ij}|l, j\rangle, \qquad i = 1, 2 \tag{48}$$

and suppose that

$$O_l^0|l, i\rangle = \sum_{j=1}^{2} g_{ij}|l, j\rangle, \qquad i = 1, 2, \tag{49}$$

where $G_{ij} = G_{ji}$ and $g_{ij} = g_{ji}$. Then the g_{ij} satisfy

$$g_{11}^2 + g_{12}^2 = G_{11}, \qquad g_{12}^2 + g_{22}^2 = G_{22}, \qquad g_{12}(g_{11} + g_{22}) = G_{12}. \tag{50}$$

Provided G_{12} and $(G_{11} - G_{22})$ do not both vanish the solutions of these equations are

$$g_{11} = x\left(\frac{G_{11}(G_{11}-G_{22})^2 + (3G_{11}-G_{22})G_{12}^2 \pm 2G_{12}^2(G_{11}G_{22}-G_{12}^2)^{1/2}}{(G_{11}-G_{22})^2 + 4G_{12}^2}\right)^{1/2} \tag{51}$$

$$g_{12} = x\left(\frac{(G_{11}+G_{22}) \mp 2(G_{11}G_{22}-G_{12}^2)^{1/2}}{(G_{11}-G_{22})^2 + 4G_{12}^2}\right)^{1/2} \tag{52}$$

$$g_{22} = x\left(\frac{G_{22}(G_{11}-G_{22})^2 + (3G_{22}-G_{11})G_{12}^2 \pm 2G_{12}^2(G_{11}G_{22}-G_{12}^2)^{1/2}}{(G_{11}-G_{22})^2 + 4G_{12}^2}\right)^{1/2}, \tag{53}$$

where $x = \pm 1$. We therefore have four different sets of solutions depending on x and the sign of $\sqrt{(G_{11}G_{22}-G_{12}^2)}$. No phase arbitrariness is left in the $|l, i\rangle$ so the g_{ij} must be uniquely soluble; knowledge of the matrix elements of $(O_l^0)^2$ alone is therefore insufficient to completely determine those of O_l^0, so further conditions must be sought. These are provided by the matrix elements of $O_{l+1}^{-1}O_{l+2}^{-1}O_l^{+2}$, which are obtained from the known actions of O_{l+1}^{-1}, O_{l+2}^{-1} and O_l^{+2} and related, using (I, 39) and the known matrix elements of Q_l^0 and $(O_l^0)^2$, to the g_{ij}. $O_{l+1}^{-1}O_{l+2}^{-1}O_l^{+2}$ has four independent matrix elements, and each of these is a linear combination of the g_{ij} (once the numerical values for the matrix elements of Q_l^0 and $(O_l^0)^2$ are used) and a constant term arising from the I_3 term. Any three of the resulting simultaneous equations in the g_{ij} enables us to determine them uniquely.

For the $l = 2$ states of (4, 2) and (6, 2), $(O_l^0)^2$ turns out to be diagonal and degenerate, and in these cases equations (50) reduce to

$$g_{11}^2+g_{12}^2 = g_{22}^2+g_{12}^2 = G_{11}, \qquad g_{12}(g_{11}+g_{22}) = 0 \tag{54}$$

so $g_{11}^2 = g_{22}^2$. If $g_{11} = g_{22}$ then $g_{12} = 0$, but if $g_{11} = -g_{22}$ then g_{12} need not be zero and (54) are inadequate for the solution of the g_{ij}. The latter was found to be the case for both above pairs of states, so that although in both cases $(O_l^0)^2$ is diagonal on the $|2, 1\rangle$ and $|2, 2\rangle$, neither states are eigenvectors of O_l^0 itself. For (4, 2) O_2^0 is completely off-diagonal and its sign is determined by considering the value of $\langle 2, 2|O_3^{-1}O_4^{-1}O_2^{+2}|2, 1\rangle$, whereas for (6, 2) it was found, by considering any three matrix elements of $O_3^{-1}O_4^{-1}O_2^{+2}$, that O_2^0 has both diagonal and off-diagonal matrix elements. Once the g_{ij} have been found, the eigenvalues of O_l^0 are obtained by the usual method.

4. Complete analysis of the $(p, 0)$ representations

It can be shown (Hughes 1973b) that for the SU(3) representations realized by wavefunctions of the three-dimensional harmonic oscillator, the invariants satisfy the relationship $6I_3 = I_2(4I_2+1)^{1/2}$. From this it is easy to deduce that the representations were the $(p, 0)$, for which $I_2 = \frac{1}{9}p(p+3)$ and $I_3 = \frac{1}{162}p(p+3)(2p+3)$, and which have dimension $\frac{1}{2}(p+1)(p+2)$.

Since no l degeneracy occurs in these representations, they can be fully analysed by the techniques of this paper. We give here an inductive proof of the following results: (*a*) l takes on the non-degenerate values $p, p-2, p-4, \ldots, 0$ or 1, depending on whether p is even or odd; (*b*) for l having any of the above values, the actions of the various operators on $|l\rangle$ are

$$O_{l+2}^{-2}O_l^{+2}|l\rangle = 24(l+1)^4(l+2)^4(p-l)(p+l+3)|l\rangle \tag{55}$$

$$O_{l+1}^{-1}O_l^{+1}|l\rangle = O_{l-1}^{+1}O_l^{-1}|l\rangle = 0 \tag{56}$$

$$O_{l-2}^{+2}O_l^{-2}|l\rangle = 24l^4(l-1)^4(p+l+1)(p-l+2)|l\rangle \tag{57}$$

$$O_l^{+2}|l\rangle = 2\sqrt{6}(l+1)^2(l+2)^2\left(\frac{(2l+1)(p-l)(p+l+3)}{2l+5}\right)^{1/2}|l+2\rangle \tag{58}$$

$$O_l^{+1}|l\rangle = O_l^{-1}|l\rangle = 0 \tag{59}$$

$$O_l^{-2}|l\rangle = 2\sqrt{6}l^2(l-1)^2\left(\frac{(2l+1)(p+l+1)(p-l+2)}{2l-3}\right)^{1/2}|l-2\rangle \tag{60}$$

$$O_l^0|l\rangle = \sqrt{6l(l+1)(2p+3)}|l\rangle \tag{61}$$

$$Q_l^0|l\rangle = 2l(l+1)(4p^2+12p-6l^2-6l-27)|l\rangle. \tag{62}$$

The first step in the proof is to check that these formulae are valid for the highest l value, and this is easily done by substituting $l = p$ in them and comparing the results with the corresponding formulae of §§ 2 and 3 with $q = 0$. The fact that $O_p^{-1}|p\rangle = 0$ shows that $|p-1\rangle$ does not occur so that $l = p-2$ is non-degenerate.

Now assume the formulae valid for some l differing from p by an even integer and deduce their validity for $|l-2\rangle$. Firstly, since $|l+1\rangle$ is assumed not to exist and $O_l^{-1}|l\rangle = 0$ it follows that $|l-1\rangle$ does not exist, and since l is assumed to be non-degenerate so then is $l-2$. It therefore follows that:

$$\langle l-2|O_l^{-2}O_{l-2}^{+2}|l-2\rangle = \langle l|O_{l-2}^{+2}O_l^{-2}|l\rangle$$

so

$$O_l^{-2}O_{l-2}^{+2}|l-2\rangle = 24l^4(l-1)^4(p+l+1)(p-l+2)|l+2\rangle$$

which is just (55) with l replaced by $l-2$. The formula for $O_{l-2}^{+2}|l-2\rangle$ is then easily obtained and found to agree with (58). Also, since $|l-1\rangle$ does not exist, $O_{l-2}^{+1}|l-2\rangle$ and $O_{l-1}^{-1}O_{l-2}^{+1}|l-2\rangle$ both vanish. Knowing the actions of $O_{l-1}^{-1}O_{l-2}^{+1}$ and $O_l^{-2}O_{l-2}^{+2}$ on $|l-2\rangle$ enables us, using (I, 45) to (I, 48), to calculate the actions of $O_{l-3}^{+1}O_{l-2}^{-1}$, $O_{l-4}^{+2}O_{l-2}^{-2}$, Q_{l-2}^0 and $(O_{l-2}^0)^2$ on $|l-2\rangle$, the appropriate sign for $O_{l-2}^0|l-2\rangle$ being then determined using $O_{l-1}^{-1}O_l^{-1}O_{l-2}^{+2}|l-2\rangle = 0$. Also, knowing $O_{l-4}^{+2}O_{l-2}^{-2}|l-2\rangle$ enables us to calculate $O_{l-2}^{-2}|l-2\rangle$. The results obtained are found to be those given in (*b*) above with l replaced by $l-2$. The validity of the formulae of (*b*) for all l therefore follows by induction, the fact that l has minimum value 0 or 1 being a consequence of the presence of the factors l and $(l-1)$ in equation (57).

From (61) we see that if $|0\rangle$ occurs then $O_0^0|0\rangle = 0$. This is true of any representation (p, q) containing a $|0\rangle$, as can easily be checked by using (I, 36) and (I, 38) to express $(O_l^0)^2$ in terms of $O_{l-1}^{+1}O_l^{-1}$ and $O_{l-2}^{+2}O_l^{-2}$ and noting that both these operators vanish when $l = 0$. Finally, the analysis of the contragredient representation (p, p) follows from that of $(p, 0)$ simply by changing the sign of I_3 and O_l^0: otherwise the two sets of representations are identical.

5. Conclusion

We have seen here that the use of the l shift operators enables a solution of the state labelling problem of SU(3) in an O(3) basis to be given, any given representation (p, q) being completely analysable into orthonormal l states, and all eigenvalues of the hermitian operators O_l^0 and Q_l^0 being obtainable. The drawback of the method is that it provides an algorithm rather than closed formulae giving all matrix elements as explicit functions of p, q and l. By the very nature of the problem, however, it is quite possible that an algorithmic solution is the best that could be hoped for.

Nevertheless, closed formulae were obtained very easily for the $(p, 0)$ and there is little doubt that the same would be true of the $(p, 1)$. Of the other representations we see most hope of obtaining closed formulae for the self-contragredient representations $(p, \frac{1}{2}p)$. We hope to consider these two sets of representations in a later paper.

In this and the previous paper we have restricted our considerations to states of zero I_0 eigenvalue, as a result of which many of the formulae obtained are valid only when $m = 0$. The reduction in the algebraic computation necessary more than compensated for this slight loss of generality, especially since the eigenvalues of Q_l^0 and O_l^0 are anyway independent of m. An O(3) analysis of O(4), O(3, 1) and the euclidean group E(3) could easily be given using l shift operator techniques similar to the ones used here; since they contain no l degeneracies they would be far easier to treat than SU(3) and could be easily treated without the assumption that $m = 0$.

In this paper it was assumed that for unitary irreducible representations (p, q) of SU(3) the parameters p and q were integers satisfying $p \geqslant q \geqslant 0$. By more efficient use of the various hermiticity relations these values of p and q could have been derived although it would have been pointless to do so. However, if the generators q_i of SU(3) are replaced by iq_i the non-compact group SL(3, R), which has O(3) as a maximal compact subgroup, is obtained. For this group the invariants can still be expressed in terms of p and q by formulae (I, 1) and (I, 2) and so the pair (p, q) may still be used to label its irreducible representations. The unitary representations of SL(3, R) will, however, be infinite dimensional so p and q will not have the same values as for SU(3). By simple modifications of the formulae of I it should be possible to use the l shift operators to analyse this group with respect to O(3) and to derive, making full use of the hermiticity relations, the values of p and q specifying its unitary representations.

Techniques similar to those of these two papers can clearly also be used for larger groups. For example SU(4) contains an O(3) subgroup such that the generators, apart from the l_i, form both a five- and a seven-dimensional operator representation of O(3). It should be possible therefore to construct l shift operators O_l^0, $O_l^{\pm 1}$, $O_l^{\pm 2}$ and $\bar{O}_l^0, \bar{O}_l^{\pm 1}, \bar{O}_l^{\pm 2}, \bar{O}_l^{\pm 3}$, and use them to give a full analysis of the representations of SU(4) with respect to this O(3) subgroup.

Finally, note that all the most important representations occurring in the octet model of hadron physics have been dealt with here; these are the quark representations (1, 0) and (1, 1), the octet (2, 1), the decuplets (3, 0) and (3, 3) and the 27-dimensional representation (4, 2). It would be interesting to investigate whether the O(3) subgroup discussed here has any significance for hadrons.

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Appendix. Eigenvalues of O_l^0 and Q_l^0

(*a*) (4, 2):

l	4	3	2	0
O_l^0	0	0	$\pm 18\sqrt{70}$	0
Q_l^0	360	-1512	396 -1908	0

(*b*) (5, 2):

l	5	4	3	2	1
O_l^0	$78\sqrt{6}$	$88\sqrt{6}$	$6\sqrt{6}(11\pm\sqrt{241})$	$-66\sqrt{6}$	$22\sqrt{6}$
Q_l^0	432	-2456	$24(37\pm2\sqrt{3529})$	-1812	196

(*c*) (6, 2):

l	6	5	4	3	2	0
O_l^0	$210\sqrt{6}$	$234\sqrt{6}$	$320\sqrt{6}$ $-44\sqrt{6}$	0	$\pm6\sqrt{1158}$	0
Q_l^0	1428	-4068	$-8(133\pm$ $6\sqrt{14569})$	-1512	1452 -2964	0

(*d*) (2, 1):

l	2	1
O_l^0	0	0
Q_l^0	-108	-372

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